UNIVERSITY OF KERALA
Model Question Paper
First Degree Programme
Semester VI Core Course
MM: 1641 Real Analysis II

Time: 3 Hours **Maximum Marks: 80**

Section I
All the first 10 questions are compulsory. Each carries 1 mark.

1. Is signum function continuous at $x = 0$?
2. Determine the points of continuity of the function $f(x) = x[x]$, where [] denote the greatest integer function.
3. Give an example of a function $f : [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$ but $|f|$ is continuous on $[0,1]$.
4. Give an example of a function which is monotone but not continuous.
5. Find $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$.
6. Is the function $f(x) = |x| + |x+1|$ from $\mathbb{R} \to \mathbb{R}$ differentiable?
7. Find the points of relative extreme for $f(x) = |x^2 - 1|$ of $-4 \leq x \leq 4$.
8. State Taylor's theorem.
9. Give an example of a function which is not Riemann integrable.
10. Define a null set.

Section II
Answer any 8 questions from this section.
Each question carries 2 marks

11. Show that the Dirchilet's function is not continuous at any point of $\mathbb{R}$.
12. Show that absolute value function $f(x) = |x|$ is continuous at every point of $\mathbb{R}$.
13. Give an example of functions f and g that are both discontinuous at a point $c \in \mathbb{R}$ such that the sum $f + g$ is continuous at c.
14. Discuss the continuity of the function $F : \mathbb{R} \to \mathbb{R}$, at $x = 0$ defined by

$$F(x) = \begin{cases} 0 & \text{if } x = 0. \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

15. Discuss the continuity of the cosine function on $\mathbb{R}$.

16. Discuss the differentiability on $\mathbb{R}$, of the function

$$F(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

17. Find $\lim\limits_{x \to \infty} e^{-x} x^2$.

18. Show that if $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

19. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) and that $f'(x) = 0$ for all $x \in (a, b)$. Show that f is constant on I.

20. If $f, g \in R[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, Show that $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

21. State Lebesgue integrability criterion. Use it to show that every step function on $[a, b]$ is reimann integrable.

22. If $f \in R[a, b]$, show that $|f| \in R[a, b]$ and $\displaystyle|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx \leq M(b-a)$, where $|f| \leq M$ for all $x \in [a, b]$.

Section III
Answer any 6 questions from this section.
Each question carries 4 marks.

23. State and prove boundedness theorem.

24. State and prove Bolzano intermediate value theorem.

25. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function on I, show that $f(I)$ is a closed bounded interval.

26. Establish chain rule of differentiation.

27. State and prove mean value theorem.

28. Show that $1 - \dfrac{x^2}{2} \leq \cos x$ for all $x \in \mathbb{R}$.

29. If $f \in R[a, b]$, show that f is bounded on $[a, b]$.

30. Show that if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in R[a, b]$.

31. State and prove Fundamental theorem of Calculus (First form).

Section IV
Answer any 2 questions from this section.
Each question carries 15 marks.

32. **(a)** State and prove Maximum Minimum theorem.
    (b) State and prove location of roots theorem.

33. **(a)** State and prove Caratheodary theorem.

(b) Prove that sum, product and quotient of functions differentiable at a point $c \in I$ are differentiable at c.

34. **(a)** Establish Cauchy criterion for riemann integrability.

    (b) State and prove squeeze theorem.

35. **(a)** Let $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$. Prove that If $f \in R[a, b]$ if and only if its restriction to $[a, c]$ and $[c, b]$ are both riemann integrable. Also show that $\int_a^b f = \int_a^c f + \int_c^b f$.

    (b) If $f,\ g \in R[a, b]$, show that If $fg \in R[a, b]$.

    (c) Discuss the method of integration by parts for Riemann integrals.