Section I
All the first 10 questions are compulsory. Each carries 1 mark.

1. Let \( \phi : S_3 \rightarrow \mathbb{Z}_2 \) be a homomorphism defined by
   \[
   \phi(\sigma) = \begin{cases} 
   0 & \text{if } \sigma \text{ is an even permutation.} \\
   1 & \text{if } \sigma \text{ is an odd permutation.}
   \end{cases}
   \]
   Compute \( \ker \phi \).

2. Find the order of \( \mathbb{Z}_6/\langle 3 \rangle \).

3. Compute \((11)(-4)\) in the ring \( \mathbb{Z}_{15} \).

4. Let \( p \) be a prime. Find number of 0 divisors of \( \mathbb{Z}_p \).

5. Give an example of an integral domain.

6. Find the characteristic of the ring \( \mathbb{Q} \).

7. State little theorem of Fermat.

8. Compute \( \phi(21) \).

9. Find all units in the ring \( \mathbb{Z} \times \mathbb{Z} \).

10. How many homomorphisms are there of \( \mathbb{Z} \) into \( \mathbb{Z}_2 \)?

Section II
Answer any 8 questions from this section.
Each question carries 2 marks

11. Let \( \phi \) be a homomorphism of a group \( G \) into a group \( G' \) and \( H \) be a subgroup of \( G \). Prove that \( \phi(H) \) is a subgroup of \( G' \).

12. Let \( G \) be a group, and let \( g \in G \). Let \( \phi_g : G \rightarrow G \) be defined by \( \phi_g(x) = gxg^{-1} \) for \( x \in G \). Prove that \( \phi_g \) is a homomorphism.

13. Let \( H \) be subgroup of an abelian group \( G \). Prove that \( H \) is a normal subgroup of \( G \).

14. Prove that a factor group of a cyclic group is cyclic.

15. Compute the factor group \( \mathbb{Z}_4 \times \mathbb{Z}_6/\langle(0, 1)\rangle \).

16. Find the order of \( 26 + \langle 12 \rangle \) in \( \mathbb{Z}_{60}/\langle 12 \rangle \).
17. Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$.

18. Show that the matrix ring $M_2(\mathbb{Z}_2)$ has divisors of zero.

19. Find all solutions of $x^2 + 2x + 4 = 0$ in $\mathbb{Z}_6$.

20. Let $R$ be a commutative ring with characteristic 4. Compute and simplify $(a + b)^4$ for $a, b \in R$.

21. Find the units of $\mathbb{Z}_{14}$.

22. Show that an intersection of ideals of a ring $R$ is again an ideal of $R$.

Section III

Answer any 6 questions from this section.
Each question carries 4 marks.

23. Prove that a group homomorphism $\phi : G \to G$ is a one to one map if and only if $\ker(\phi) = \{e\}$.

24. Let $H$ be a normal subgroup of a group $G$. Prove that the following are equivalent.
   (a) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.
   (b) $ghg^{-1} = H$ for all $g \in G$.
   (c) $gH = Hg$ for all $g \in G$.

25. Is the converse of Lagrange’s theorem true? Justify.

26. Consider the matrix ring $M_2(\mathbb{Z}_2)$.
   (a) Find the order of the ring.
   (b) List all units in the ring.

27. Let $a \in \mathbb{Z}$ and $p$ be a prime not dividing $a$. Show that $a^{p-1} \equiv 1 \pmod{p}$.

28. Compute the remainder of $8^{103}$ when divided by 13.

29. An element $a$ of a ring $R$ is idempotent if $a^2 = a$
   (a) Show that the set of all idempotent elements of a commutative ring is closed under multiplication.
   (b) Find all idempotents in the ring $\mathbb{Z}_6 \times \mathbb{Z}_{12}$.

30. Show that cancellation laws hold in a ring $R$ if and only if $R$ has no zero divisors.

31. Let $\phi : G \to G'$ be a group homomorphism. Prove that $\ker(\phi)$ is a normal subgroup of $G$.

Section IV

Answer any 2 questions from this section.
Each question carries 15 marks.
32. Let \( \phi : G \to G' \) be a group homomorphism with kernel \( H \). Prove that the cosets of \( H \) form a factor group \( G/H \), where \( (aH)(bH) = (ab)H \). Also the map \( \mu : G/H \to \phi[G] \) defined by \( \mu(aH) = \phi(a) \) is an isomorphism. Both coset multiplication and \( \mu \) are well defined, independent of the choices of \( a \) and \( b \) from the cosets.

33. Let \( GL(n, \mathbb{R}) \) be the multiplicative group of all invertible \( n \times n \) matrices and \( \mathbb{R}^* \) be multiplicative group of nonzero real numbers. Let \( \phi : GL(n, \mathbb{R}) \to \mathbb{R}^* \) be given by \( \phi(A) = \det A \), the determinant of \( A \), for \( A \in GL(n, \mathbb{R}) \). Prove that

(a) \( \phi \) is a homomorphism.

(b) \( n \times n \) matrices with determinant 1 form a normal subgroup of \( GL(n, \mathbb{R}) \).

(c) \( n \times n \) matrices with determinant \( \pm 1 \) form a normal subgroup of \( GL(n, \mathbb{R}) \).

34. Prove that

(a) Every field is an integral domain.

(b) Every finite integral domain is a field.

(c) \( \mathbb{Z}_p \) is a field, for a prime \( p \).

35. Prove that

(a) The set \( G_n \) of nonzero elements of \( \mathbb{Z}_n \) that are not 0 divisors forms a group under multiplication modulo \( n \).

(b) \( a^{\varphi(n)} \equiv 1(\mod n) \), where \( a \) is an integer relatively prime to \( n \).

(c) Find all solutions of the congruence \( 15x \equiv 27(\mod 18) \).