KERALA UNIVERSITY
Model Question Paper- MSc Examination
Branch: Mathematics
MM 234- Differential Geometry

Time: 3 hours | Max. Marks: 75

Part A
Answer any 5 questions from among the questions from 1 to 8.
Each question carries 3 marks

1. Show that graph of any function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$.

2. Find and sketch the gradient field of $f(x_1, x_2) = x_1 - x_2^2$.

3. Describe the spherical image, when $n = 1, 2$, of the surface $-x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 0,\ x_1 > 0$, oriented by the outward normal.

4. Let S be an n-plane $a_1x_1 + \cdots + a_{n+1}x_{n+1} = b$, let $p, q \in S$ and $\mathbf{v} = (p, v) \in S_p$. If α is any parametrized curve in S from p to q, find the parallel transport along α of $\mathbf{v}$.

5. Compute $\nabla_{\mathbf{v}} f$, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $\mathbf{v} \in R^3_p,\ p \in R^3$ is $f(x_1, x_2, x_3) = x_1x_2x_3^2$, $\mathbf{v} = (1, 1, 1, a, b, c)$.

6. If $\mathbf{X}$ and $\mathbf{Y}$ are smooth vector fields on an n-surface S with $\mathbf{v} \in S_p,\ p \in S$, then show that $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y}) = (\nabla_{\mathbf{v}}\mathbf{X}) \cdot \mathbf{Y}(p) + \mathbf{Y}(p) \cdot (\nabla_{\mathbf{v}}\mathbf{Y})$.

7. Show that the integral of an exact 1-form over a closed curve is zero.

8. Evaluate the normal curvature of n-sphere S of radius r at $p \in S$ in the direction of $\mathbf{v} \in S_p$.

$(5 \times 3 = 15)$

Part B
Answer all questions from 9 to 13.
Each question carries 12 marks

9. A (a). State and prove existence and uniqueness of maximal integral curve of a smooth vector field $\mathbf{X}$ on an open set $U \subseteq \mathbb{R}^{n+1}$ and passing through $p \in U$.

   (b). Sketch the level sets $f^{-1}(-1),\ f^{-1}(0)$ for $f(x_1, x_2, \cdots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2,\ n = 1$.

OR

B (a). Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and $f : U \longrightarrow \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

(b). Find the integral curve through (a, b) of the vector field on $\mathbb{R}^2$ given by $\mathbf{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (x_2, -x_1)$.

10. A (a). Show that each connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

(b). Show that the maximum and minimum values of $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, where $a, b, c \in \mathbb{R}$, on the circle $x_1^2 + x_2^2 = 1$ are the eigenvalues of a symmetric 2×2 matrix.

OR

B (a). State and prove existence and uniqueness of maximal integral curve of a smooth tangent vector field $\mathbf{X}$ on an n-surface $S \subseteq \mathbb{R}^{n+1}$ and through $p \in S$.

(b). State and prove Lagrange's Multiplier theorem.

11. A (a). Let S be the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in $\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$ for some $a, b, c, d \in \mathbb{R}$.

(b). Let $S \subseteq \mathbb{R}^{n+1}$ be an n-surface, $p, q \in S$ and α be a piecewise smooth parametrized curve from p to q. Show that parallel transport along α is a vector space isomorphism which preserves dot products.

OR

B (a). State and prove existence and uniqueness of maximal geodesic in an n-surface S passing through $p \in S$ with initial velocity $\mathbf{v} \in S_p$.

(b). State and prove any four properties of Levi-Civita parallelism.

12. A (a). Show that the Weingarten map of an n-surface S at $p \in S$ is self adjoint.

(b). Find the global parametrization of $ax_1 + bx_2 = c, \; (a, b) \neq (0, 0)$, oriented by the outward normal. Also find the curvature.

OR

B (a). Show that local parametrizations of plane curves are unique upto reparametrization.

(b). Compute the weingarten map of the n-sphere $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$, oriented by the inward normal.

13. A (a). Show that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$, there exists a point p such that the second fundamental form at p is definite.

(b). Find the principal curvatures and the Guassian curvature of the hyperboloid $-x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$, oriented by the outward normal, at $p = (0, 0, 1)$.

OR

B (a). For each 1-form ω on U(U open in $\mathbb{R}^{n+1}$) there exist unique functions $f_i : U \longrightarrow \mathbb{R}(i \in \{1, 2, \cdots, n+1\})$ such that $\omega = \sum_{i=1}^{n+1} f_i dx_i$. Moreover, ω is smooth if and only if each f_i is smooth.

(b). Find the Guassian curvature of a cylinder over a plane curve.

$(5 \times 12 = 60)$