UNIVERSITY OF KERALA
Model Question Paper
First Degree Programme
Semester VI Core Course
MM: 1644 Abstract Algebra II

Time: 3 Hours **Maximum Marks: 80**

Section I

All the first 10 questions are compulsory. Each carries 1 mark.

1. Let $\phi : S_3 \to \mathbb{Z}_2$ be a homomorphism defined by

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is an even permutation.} \\ 1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

   Compute $ker\phi$.

2. Find the order of $\mathbb{Z}_6/\langle 3\rangle$.
3. Compute $(11)(-4)$ in the ring $\mathbb{Z}_{15}$.
4. Let p be a prime. Find number of 0 divisors of $\mathbb{Z}_p$.
5. Give an example of an integral domain.
6. Find the characteristic of the ring $\mathbb{Q}$.
7. State little theorem of Fermat.
8. Compute $\varphi(21)$.
9. Find all units in the ring $\mathbb{Z} \times \mathbb{Z}$.
10. How many homomorphisms are there of $\mathbb{Z}$ into $\mathbb{Z}_2$?

Section II

Answer any 8 questions from this section.
Each question carries 2 marks

11. Let ϕ be a homomorphism of a group G into a group G' and H be a subgroup of G. Prove that $\phi(H)$ is a subgroup of G'.
12. Let G be a group, and let $g \in G$. Let $\phi_g : G \to G$ be defined by $\phi_g(x) = gxg^{-1}$ for $x \in G$. Prove that ϕ_g is a homomorphism.
13. Let H be subgroup of an abelian group G. Prove that H is a normal subgroup of G.
14. Prove that a factor group of a cyclic group is cyclic.
15. Compute the factor group $\mathbb{Z}_4 \times \mathbb{Z}_6/\langle(0, 1)\rangle$.
16. Find the order of $26 + \langle 12\rangle$ in $\mathbb{Z}_{60}/\langle 12\rangle$.

17. Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$.

18. Show that the matrix ring $M_2(\mathbb{Z}_2)$ has divisors of zero.

19. Find all solutions of $x^2 + 2x + 4 = 0$ in $\mathbb{Z}_6$.

20. Let R be a commutative ring with characteristic 4. Compute and simplify $(a + b)^4$ for $a, b \in R$.

21. Find the units of $\mathbb{Z}_{14}$.

22. Show that an intersection of ideals of a ring R is again an ideal of R.

Section III

Answer any 6 questions from this section.

Each question carries 4 marks.

23. Prove that a group homomorphism $\phi : G \to G$ is a one to one map if and only if $ker(\phi) = \{e\}$.

24. Let H be a normal subgroup of a group G. Prove that the following are equivalent.

    (a) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

    (b) $ghg^{-1} = H$ for all $g \in G$.

    (c) $gH = Hg$ for all $g \in G$.

25. Is the converse of Lagrange's theorem true? Justify.

26. Consider the matrix ring $M_2(\mathbb{Z}_2)$.

    (a) Find the order of the ring.

    (b) List all units in the ring.

27. Let $a \in \mathbb{Z}$ and p be a prime not dividing a. Show that $a^{p-1} \equiv 1(mod\ p)$.

28. Compute the remainder of 8^{103} when divided by 13.

29. An element a of a ring R is idempotent if $a^2 = a$

    (a) Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

    (b) Find all idempotents in the ring $\mathbb{Z}_6 \times \mathbb{Z}_{12}$.

30. Show that cancellation laws hold in a ring R if and only if R has no zero divisors.

31. Let $\phi : G \to G'$ be a group homomorphism. Prove that $ker(\phi)$ is a normal subgroup of G.

Section IV

Answer any 2 questions from this section.

Each question carries 15 marks.

32. Let $\phi : G \to G'$ be a group homomorphism with kernel H. Prove that the cosets of H form a factor group G/H, where $(aH)(bH) = (ab)H$. Also the map $\mu : G/H \to \phi[G]$ defined by $\mu(aH) = \phi(a)$ is an isomorphism. Both coset multiplication and μ are well defined, independent of the choices of a and b from the cosets.

33. Let $GL(n, \mathbb{R})$ be the multiplicative group of all invertible $n \times n$ matrices and $\mathbb{R}^*$ be multiplicative group of nonzero real numbers. Let $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$ be given by $\phi(A) = detA$, the determinant of A, for $A \in GL(n, \mathbb{R})$. Prove that

    (a) ϕ is a homomorphism.

    (b) $n \times n$ matrices with determinant 1 form a normal subgroup of $GL(n, \mathbb{R})$.

    (c) $n \times n$ matrices with determinant ± 1 form a normal subgroup of $GL(n, \mathbb{R})$.

34. Prove that

    (a) Every field is an integral domain.

    (b) Every finite integral domain is a field.

    (c) $\mathbb{Z}_p$ is a field, for a prime p.

35. Prove that

    (a) The set G_n of nonzero elements of $\mathbb{Z}_n$ that are not 0 divisors forms a group under multiplication *modulo* n.

    (b) $a^{\varphi(n)} \equiv 1(mod\ n)$, where a is an integer relatively prime to n.

    (c) Find all solutions of the congruence $15x \equiv 27(mod\ 18)$.