KERALA UNIVERSITY

Model Question Paper- M. Sc. Examination 2020 admission onwards

Branch : Mathematics

MM 244- Number Theory

Time: 3 hours | Max. Marks:75

Part A

Answer any 5 questions from among the questions 1 to 8

Each question carries 3 marks

1. Prove that $\phi(n) > \frac{n}{6}$ for all n with at most 8 distinct prime factors.
2. Find all integers n such that $\phi(n) = \frac{n}{2}$
3. Prove that there are infinitely many primes of the form $4n - 1$
4. Prove that 5 is a quadratic residue of an odd prime p if $p \equiv \pm 1$ mod 10, and that 5 is a nonresidue if $p \equiv \pm 3$ mod 10.
5. Determine whether 888 is a quadratic residue or nonresidue of the prime 1999.
6. If p is an odd prime and $\alpha \geq 1$ then prove that there exist odd primitive roots p^α.
7. Prove that m is prime if and only if $exp_m(a) = m - 1$ for some a
8. Let g be a primitive root of an odd prime p. Prove that $-g$ is also a primitive root of p if $p \cong 1$ mod 4, but that $exp_p(-g) = \frac{p-1}{2}$ if $p \cong 3$ mod 4. $5 \times 3 = 15$

Part B

Answer all questions from 9 to 13

Each question carries 12 marks

9. A. i. Prove that $\phi(n) = n\prod_{p|n}(1 - \frac{1}{p})$ $\;for\;$ $n \geq 1$ [6 Marks]

   ii. Prove that for $n \geq 1$ $\sum_{d|n} \Lambda(d) = logn$ [6 marks]

   OR

   B. Prove that the set of all arithmetical functions f with $f(1) \neq 0$ forms an abelian group with respect to the Dirchlet convolution. [12 marks]

10. A. Prove that a finite abelian group G of order n has exactly n distinct characters. [12 Marks]

    OR

    B. i. Let χ be any real-valued character $mod\ k$ and let $A(n) = \sum_{d|n} \chi(d)$. Prove that $A(n) \geq 0$ for all n and $A(n) \geq 1$ if n is a square.[4 marks]

ii. For any real-valued nonprincipal character χ mod k, let $A(n) = \sum_{d|n} \chi(d)$ and $B(x) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}}$, then prove the following

A. $B(x) \to \infty \quad as \quad x \to \infty$

B. $B(x) = 2\sqrt{x}\, L(1, \chi) + O(1)$ for all $x \geq 1$, therefore $L(1, \chi) \neq 0$ [8 marks]

11. A. For $x < 1$ and $\chi \neq \chi_1$ Prove that $\sum_{p \leq x} \frac{\chi(p) logp}{p} = -L'(1, \chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} + O(1)$

[12 Marks]

OR

B. i. For $x > 1$ Prove that $\sum_{p \leq x,\ p \cong 1 (modk)} \frac{logp}{p} = \frac{1 - N(k)}{\psi(k)} logx + O(1)$ [6 Marks]

ii. If $\chi \neq \chi_1$ and $L(1, \chi) = 0$ Prove that $L'(1, \chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} = logx + O(1)$

[6 Marks]

12. A. i. Prove that for every odd prime p, $(-1|p) = (-1)^{\frac{p-1}{2}}$ and $(2|p) = (-1)^{\frac{p^2-1}{8}}$ [9 Marks]

ii. Evaluate (780/1001) [3 Marks]

OR

B. State and prove Gauss Lemma.[12 Marks]

13. A. Let p be an odd prime and let d be any positive divisor of $p - 1$. Prove that in every reduced residue system mod p, there exactly $\phi(d)$ numbers a such that $exp_p(a) = d$ [12 Marks]

OR

B. i. Let P be an odd prime, Prove that if g is a primitive root mod p, then g is also a primitive root mod p^{α} for all $\alpha \geq 1$ if and only if $g^{p-1} \neq 1 (mod\ p^2)$ [10 Marks]

ii. Prove that m is prime if and only if $exp_m(a) = m - 1$ for some a.[2 Marks]

. $5 \times 12 = 60$